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On Laplace Transforms Near the Origin

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Abstract. Let $f(t)$ be locally integrable on $[0, \infty)$ and let $L\{f\}(s)$ denote the Laplace transform of $f(t)$. In this note, we prove that if $f(t) \sim t^{-\beta} \Sigma_{n=0}^{\infty} a_n (\log t)^{-n}$ as $t \to \infty$, where $0 \leqslant \operatorname{Re} \beta < 1$, then $L\{f\}(s) \sim s^{\beta - 1} \Sigma_{n=0}^{\infty} c_n (\log 1/s)^{-n}$ as $s \to 0$ in $|\arg s| \leqslant \pi/2 - \Delta$, the c_n being constants.

1. Introduction. Let $f(t)$ be locally integrable on $[0, \infty)$ and let $L\{f\}$ denote the Laplace transform of $f(t)$. That is,

$$L\{f\} = \int_0^\infty f(t) e^{-st}\, dt, \tag{1.1}$$

whenever the integral on the right converges. In a recent paper, Handelsman and Lew [2] have studied the asymptotic behavior of $L\{f\}$ as $s \to 0$, when $f(t)$ satisfies

$$f(t) \sim \exp(-ct^p) \sum_{m,n=0}^{\infty} c_{mn} t^{r_m} (\log t)^n \quad \text{as } t \to \infty, \tag{1.2}$$

where $p > 0$, $\operatorname{Re} c \geqslant 0$, $\operatorname{Re} r_m \downarrow -\infty$ as $m \to \infty$, and the set $\{n\colon c_{mn} \neq 0\}$ is finite for each m. In this note, we consider the case

$$f(t) \sim t^{-\beta} \sum_{n=0}^{\infty} a_n (\log t)^{-n} \quad \text{as } t \to \infty, \tag{1.3}$$

where $0 \leqslant \operatorname{Re} \beta < 1$. Our result will complement that of Handelsman and Lew.

2. Main Theorem. For convenience, we introduce the notation

$$L_c\{f\} = \int_c^\infty f(t) e^{-st}\, dt \tag{2.1}$$

where c is a fixed real number > 1. In [3], it was shown that for any complex number β with $\operatorname{Re} \beta < 1$,

$$L_c\{t^{-\beta} (\log t)^{-n}\} \sim s^{\beta - 1} \left(\log \frac{1}{s}\right)^{-n} \sum_{r=0}^{\infty} \binom{-n}{r} \Gamma^{(r)}(1 - \beta) \left(\log \frac{1}{s}\right)^{-r} \tag{2.2}$$

as $s \to 0$ in $S(\Delta)$, where

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$$S(\Delta) = \{s\colon |\arg s| \leqslant \pi/2 - \Delta\}. \tag{2.3}$$

Let

$$c_n = \sum_{r=0}^{n} a_{n-r} \binom{r-n}{r} \Gamma^{(r)}(1-\beta). \tag{2.4}$$

We are now ready to state and prove our main result.

THEOREM. *If $f(t)$ is locally integrable on* $[0, \infty)$ *and satisfies* (1.3), *then as $s \to 0$ in $S(\Delta)$*

$$L\{f\} \sim s^{\beta-1} \sum_{n=0}^{\infty} c_n \left(\log \frac{1}{s}\right)^{-n}, \tag{2.5}$$

where the coefficients c_n are given in (2.4).

Proof. For any $c > 1$,

$$L\{f\} = L_c\{f\} + \int_0^c f(t)e^{-st}\,dt = L_c\{f\} + O(1) \tag{2.6}$$

as $s \to 0$ in $S(\Delta)$.

Writing

$$f(t) = \sum_{n=0}^{N} a_n t^{-\beta}(\log t)^{-n} + R_N(t) \tag{2.7}$$

gives

$$L_c\{f\} = \sum_{n=0}^{N} a_n L_c\{t^{-\beta}(\log t)^{-n}\} + r_N, \tag{2.8}$$

where

$$r_N = \int_c^\infty R_N(t)e^{-st}\,dt. \tag{2.9}$$

From (1.3), it follows that there are constants $K > 0$ and $c > 1$ such that

$$|R_N(t)| \leqslant K|t^{-\beta}(\log t)^{-N-1}| \quad \text{for } t \geqslant c. \tag{2.10}$$

Hence, by (2.2),

$$\begin{aligned} |r_N| &\leqslant K\int_c^\infty t^{-\gamma}(\log t)^{-N-1}e^{-\sigma t}\,dt \\ &= O(\sigma^{\gamma-1}(\log \sigma)^{-N-1}) \quad \text{as } \sigma \to 0, \end{aligned} \tag{2.11}$$

where $\gamma = \operatorname{Re}\beta$ and $\sigma = \operatorname{Re} s$. Since $|s| \sin\Delta \leqslant \sigma \leqslant |s|$ for any $s \in S(\Delta)$, (2.11) is equivalent to

$$r_N = O(|s|^{\gamma-1}(\log|s|)^{-N-1}) = O(s^{\beta-1}(\log s)^{-N-1}) \tag{2.12}$$

as $s \to 0$ in $S(\Delta)$. Coupling the results (2.8) and (2.12), we obtain

$$L_c\{f\} = \sum_{n=0}^{N} a_n L_c\{t^{-\beta}(\log t)^{-n}\} + O(s^{\beta-1}(\log s)^{-N-1}) \tag{2.13}$$

as $s \to 0$ in $S(\Delta)$. Since the O-term in (2.6) may be included in that of (2.12), (2.13) implies

$$L\{f\} = \sum_{n=0}^{N} a_n L_c\{t^{-\beta}(\log t)^{-n}\} + O(s^{\beta-1}(\log s)^{-N-1}) \tag{2.14}$$

as $s \to 0$ in $S(\Delta)$. In view of (2.2), each term in (2.14) can be expanded in powers of $(\log 1/s)^{-1}$. Hence, by regrouping the terms, we have for any $N \geqslant 0$

$$L\{f\} = s^{\beta-1}\left[\sum_{n=0}^{N} c_n\left(\log \frac{1}{s}\right)^{-n} + O((\log s)^{-N-1})\right] \tag{2.15}$$

as $s \to 0$ in $S(\Delta)$. This completes the proof of our theorem.

3. An Example. The Ramanujan function is defined by

$$N(s) = \int_0^\infty \frac{e^{-st}}{t\{\pi^2 + (\log t)^2\}}\, dt. \tag{3.1}$$

Recently, Bouwkamp [1] obtained the asymptotic expansion

$$N(s) \sim \sum_{n=0}^{\infty} c_n (\log s)^{-n-1} \quad \text{as } s \to \infty, \tag{3.2}$$

where the coefficients were determined by the generating function

$$\frac{1}{\Gamma(1-x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}. \tag{3.3}$$

Our aim is to find the asymptotic behavior of $N(s)$ as $s \to 0$.

Integrating by parts, we obtain from (3.1)

$$N(s) = \frac{1}{2} + \frac{s}{\pi}\int_0^\infty \tan^{-1}\left(\frac{1}{\pi} \log t\right) e^{-st}\, dt. \tag{3.4}$$

The function $\tan^{-1}(\log t/\pi)$ has the convergent expansion

$$\tan^{-1}\left(\frac{1}{\pi} \log t\right) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{\pi}{\log t}\right)^{2n+1}, \tag{3.5}$$

for $t > e^{\pi}$. Hence, the conditions of the theorem are trivially satisfied and we have

$$N(s) \sim 1 - \sum_{\nu=0}^{\infty} a_\nu \left(\log \frac{1}{s}\right)^{-\nu-1} \tag{3.6}$$

as $s \to 0$ in $S(\Delta)$, where

$$a_\nu = \sum_{2n+r=\nu} \frac{(-1)^n \pi^{2n}}{2n+1}\binom{-2n-1}{r}\Gamma^{(r)}(1). \tag{3.7}$$

It is interesting to note that these coefficients are precisely the ones given by Bouwkamp for the asymptotic expansion of $N(s)$ as $s \to \infty$. To see this, we recall the identity

$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$. Hence, from (3.3),

$$c_\nu = \nu! \sum_{2n+r=\nu} \frac{(-1)^n \pi^{2n}}{(2n+1)!} \frac{\Gamma^{(r)}(1)}{r!}. \tag{3.8}$$

Comparing (3.7) and (3.8), we have

$$a_\nu = (-1)^\nu c_\nu, \qquad \nu = 0, 1, 2, \cdots. \tag{3.9}$$

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